# Short wavelength approximation of a boundary integral operator for homogeneous and isotropic elastic bodies

Gregor Tanner[1] and Niels Søndergaard[2]
[1]School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom
[2]Division of Mathematical Physics, Lund Technical University, 22100 Lund, Sweden
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A short wavelength approximation of a boundary integral operator for two-dimensional isotropic and homogeneous elastic bodies is derived from first principles starting from the Navier-Cauchy equation. Trace formulas for elastodynamics are deduced connecting the eigenfrequency spectrum of an elastic body to the set of periodic rays where mode conversion enters as a dynamical feature.

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## I. INTRODUCTION

Predicting energy flow within complex structures in mechanical wave problems poses an enormous challenge to a wide variety of physical and engineering applications such as the transfer of sound [1,2], signal transfer along the ocean sound channel in underwater acoustics [3], the distribution of vibrational energy in large buildup structures such as cars or airoplanes [4], or the expansion of seismic waves in earth [5,6]. A common theme emerged in these research areas in the past decade or so linking the complexity of the wave fields to the chaotic nature of an underlying ray dynamics. It has become clear that the fluctuations in wave functions or eigenspectra of such wave chaotic systems can be accurately modeled by the statistics obtained from random matrix theory [7,8]. Likewise, a ray approach describing wave fields in terms of waves traveling along classical rays is turning into a powerful tool to implement dynamical features into a wave transport theory. Wave interference effects along classically chaotic rays lead to intricate effects such as time-reversal imaging [1] or the signal enhancement due to coherent backscattering [2,6]. Furthermore, the Green function of a complex wave system can be reconstructed from the correlations in the diffusive wave field of that system [9]. This has recently been applied to the late diffusive signal of earthquake measurements containing coherent information about the elastic response of the upper earth crust [10].

Such wave chaos concepts have been pursued in some detail in quantum theory [11–13]. A powerful tool connecting the spectrum of a quantum system with an underlying classical dynamics is semiclassical expressions for the Green function and its trace as given by Gutzwiller in the 1970s [11]. Mechanical wave problems described by scalar wave functions such as in acoustics or describing membrane vibrations can be written in terms of a semiclassical expression derived in a quantum context. The situation changes drastically when considering linear wave theory in elasticity starting from the biharmonic equation describing bending modes in plates and the Navier-Cauchy equation modeling bulk deformations in elastic, isotropic bodies to shell theories or elastic waves in anisotropic media. New features enter the underlying ray dynamics due to the vectorial nature of the wave equations and the coupling between different wave modes (such as shear and pressure waves in solids). Vital prerequisites for an emerging wave chaos theory for mechanical wave equations are solid foundations on which to build transport theories driven by an underlying chaotic ray dynamics, which are, however, in large parts still missing.

We note that the governing equations of isotropic elasticity, the Navier-Cauchy equations, are separable only for a very small set of geometries such as spherical bodies or infinitely long cylindrical waveguides. Solutions to the vast majorities of shapes including rectangular bodies can be obtained only with the help of numerical techniques such as finite element or boundary integral methods [14,15]. Purely numerical approaches are, however, severely limited by computer resources and often restricted to the low frequency regime with wavelengths only one or two orders of magnitude smaller than the typical size of the system. In the high frequency limit statistical methods such as *statistical energy analysis* [4] or *random matrix theory* [7] have proved valuable. While the former yields information about mean response signals neglecting interference effects, the latter provide answers regarding the universal part of the fluctuations in the signal not taking into account system dependent effects. Semiclassical methods offer a further alternative providing detailed information about the geometry of the problem and may prove to become an important tool in the mid to high frequency regime in which the treatment of the vibrational response of large buildup structures still poses enormous problems [16].

The need for providing foundations of such an emerging asymptotic theory in elastic wave problems prompted the work presented in this paper. Here, we derive a semiclassical expression for a boundary integral kernel for isotropic and homogeneous elastic bodies of arbitrary shape in two dimensions. From the boundary kernel, semiclassical expressions for the Green function or its trace can be deduced, a method pioneered by Bogomolny [17] in the scalar case; see also Refs. [18–20]. A trace formula for the interior problem in elasticity has been presented first in Ref. [21]; the result was obtained by way of comparison with the scalar Helmholtz equation and not derived from the governing equations. The general form of such a boundary integral operator (also called transfer operator) for elastic problems has been postulated in Ref. [22] and verified for the special case of a circular waveguide. A derivation of the transfer operator for the biharmonic equation describing the out-of-plane vibrations of plates has been obtained in Ref. [23] incorporating the coupling of flexural and boundary modes. In the short wavelength limit, the wave equation reduces again to a scalar problem with modified boundary conditions due to the exponential suppression of surface waves away from the boundary.

## II. TRANSFER OPERATOR

### A. Fundamental equations

We consider isotropic and homogeneous elastic bodies described in the frequency domain by the Navier-Cauchy equation [24]

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) + \rho \omega^2 \mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}(\mathbf{r})$ is the displacement field, $\lambda$, $\mu$ are the material-dependent Lamé coefficients, and $\rho$ is the density, which we assume to be constant. We will consider free boundary conditions here; that is, no forces act normal to the boundary; this can be expressed in terms of the traction $\mathbf{t}(\mathbf{u})$; that is,

$$\mathbf{t}(\mathbf{u}) = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma}(\mathbf{u}) = 0, \tag{2}$$

where $\hat{\mathbf{n}}$ is the normal at $\mathbf{r}$ on the boundary $\mathcal{C}$ of the elastic body; the stress tensor $\boldsymbol{\sigma}(\mathbf{u})$ is given as

$$\boldsymbol{\sigma}(\mathbf{u}) = \lambda (\boldsymbol{\nabla} \cdot \mathbf{u}) \mathbf{1} + \mu (\boldsymbol{\nabla} \otimes \mathbf{u} + \mathbf{u} \otimes \boldsymbol{\nabla}). \tag{3}$$

We make the standard Helmholtz decomposition of the displacement field $\mathbf{u}$; that is,

$$\mathbf{u} = \mathbf{u}_{\mathrm{p}} + \mathbf{u}_{\mathrm{s}} \quad \text{with} \quad \mathbf{u}_{\mathrm{p}} = \boldsymbol{\nabla} \Phi, \quad \mathbf{u}_{\mathrm{s}} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}. \tag{4}$$

The elastic potentials $\Phi$ for the pressure (or longitudinal) and $\boldsymbol{\Psi}$ for the shear (or transversal) wave component solve Helmholtz's equation

$$(\Delta + k_{\mathrm{p}}^2) \Phi = 0,$$

$$(\Delta + k_{\mathrm{s}}^2) \boldsymbol{\Psi} = 0, \tag{5}$$

with wave numbers $k_{\mathrm{p}}$ and $k_{\mathrm{s}}$, respectively. One finds the dispersion relation $k_{\mathrm{p,s}} = \omega / c_{\mathrm{p,s}}$ with wave velocities

$$c_{\mathrm{p}} = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_{\mathrm{s}} = \sqrt{\frac{\mu}{\rho}}. \tag{6}$$

In the following, we shall restrict ourselves to two-dimensional problems; that is, $\mathbf{r}, \mathbf{u}(\mathbf{r}) \in \mathbb{R}^2$ and we set $\boldsymbol{\Psi} = (0,0,\Psi)^t$. The resulting differential equations describe in-plane deformations in plates or wave propagation in bodies with fixed shape in the $xy$ plane extending to $\pm\infty$ along $z$.

### B. Boundary integral equations

#### *1. General setup*

In what follows, we will adapt the method outlined in Refs. [17,23] to the Navier-Cauchy Eq. (1). We first rewrite the boundary conditions (2) in terms of boundary integral equations and then consider the Fourier coefficients of the boundary integral functions. We start by introducing the elastic potentials in the form

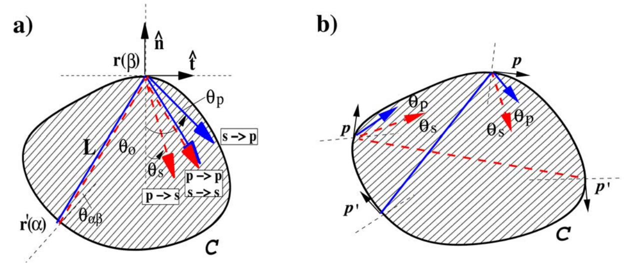


FIG. 1. (Color online) Coordinates on the boundary: (a) Position representation with path of length $L$ from $\mathbf{r}'(\alpha)$ to $\mathbf{r}(\beta)$ (here for an initial pressure wave); (b) momentum representation with shear and pressure path starting with tangential momentum $p'$ and ending with momentum $p$ on the boundary.

$$\Phi(\mathbf{r}) = \int_{\mathcal{C}} G(\mathbf{r}, \alpha; k_{\mathrm{p}}) g(\alpha) d\alpha, \tag{7}$$

$$\Psi(\mathbf{r}) = \int_{\mathcal{C}} G(\mathbf{r}, \alpha; k_{\mathrm{s}}) h(\alpha) d\alpha, \tag{8}$$

where $g$ and $h$ are yet unknown single-layer distributions on the boundary and $\alpha \in [0, L_C]$ parametrizes the boundary of length $L_C$; that is, $\mathbf{r}(\alpha) \in \mathcal{C}$; furthermore, $G(\mathbf{r}, \mathbf{r}'; k)$ is a Green function solving the inhomogeneous Helmholtz equation

$$(\Delta + k^2) G(\mathbf{r}, \mathbf{r}'; k) = \delta(\mathbf{r} - \mathbf{r}').$$

The integrals converge for $\mathbf{r}$ inside $\mathcal{C}$ and nonsingular layer distributions $g$ and $h$, and the ansatz (7) and (8) thus solves the Helmholtz equation in the interior. A convenient choice for $G(\mathbf{r}, \mathbf{r}'; k)$ is the free Green function, which in two dimensions takes the form

$$G(\mathbf{r}, \mathbf{r}', k) = \frac{1}{4i} H_0^{(1)}(k |\mathbf{r} - \mathbf{r}'|), \tag{9}$$

where $H_0^{(1)}$ is the zeroth-order Hankel function.

In a next step, it is useful to rewrite the boundary condition (2) in terms of the elastic potentials. Defining $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ as the (outward) normal and tangent vectors at the boundary point $\mathbf{r}(\beta) \in \mathcal{C}$ as indicated in Fig. 1(a), one obtains

$$\hat{\mathbf{n}} \cdot \boldsymbol{\sigma} = -\lambda \hat{\mathbf{n}} k_{\mathrm{p}}^2 \Phi + 2\mu \left[ \hat{\mathbf{n}} \frac{\partial^2}{\partial n^2} \Phi + \hat{\mathbf{t}} \frac{\partial^2}{\partial n \partial t} \Phi + \hat{\mathbf{n}} \frac{\partial^2}{\partial n \partial t} \Psi + \hat{\mathbf{t}} \frac{1}{2} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial n^2} \right) \Psi \right] = 0, \tag{10}$$

where we used $\Delta \Phi = -k_p^2 \Phi$ valid in the interior; note that all partial derivatives are understood as being taken in the interior (after a suitable continuation of the local coordinates system into the interior) and then taking the limit $\mathbf{r} \to \mathbf{r}(\beta) \in C$.

We thus need to determine derivatives of the form

$$\partial_{nn} \int_{\mathcal{C}} G(\beta, \alpha) f(\alpha) d\alpha,$$

$$\partial_{nt}\int_{\mathcal{C}} G(\beta,\alpha)f(\alpha)d\alpha,$$

$$\partial_{tt}\int_{\mathcal{C}} G(\beta,\alpha)f(\alpha)d\alpha, \tag{11}$$

with $G(\beta,\alpha)\equiv G[\mathbf{r}(\beta),\mathbf{r}'(\alpha),k]$, $f$ stands for $g$ or $h$, respectively, and the derivatives are always taken with respect to the first variable $\mathbf{r}(\beta)$ from the interior. Note that taking the limit $\mathbf{r}\to\mathbf{r}(\beta)$ and differentiating are noncommuting operations due to the logarithmic singularity of the Green function for $\beta\to\alpha$.

For a short wavelength analysis, we distinguish between long segments with $k|\mathbf{r}(\beta)-\mathbf{r}'(\alpha)|\gg 1$ and short contributions with $k|\mathbf{r}(\beta)-\mathbf{r}'(\alpha)|=\mathcal{O}(1)$; for the former, one can employ the asymptotic form of the Green function

$$G(\mathbf{r},\mathbf{r}',k)\sim\frac{1}{4i}\sqrt{\frac{2}{\pi k|\mathbf{r}-\mathbf{r}'|}}e^{i(k|\mathbf{r}-\mathbf{r}'|-\pi/4)},\quad k|\mathbf{r}-\mathbf{r}'|\to\infty, \tag{12}$$

whereas the logarithmic singularity $G(\mathbf{r},\mathbf{r}',k)\sim\frac{1}{2\pi}\ln(k|\mathbf{r}-\mathbf{r}'|)$ for $k|\mathbf{r}-\mathbf{r}'|\to 0$ calls for a separate treatment for short length contributions. We note, in particular, that one obtains from Eq. (12) in leading order,

$$\partial_n G(\beta,\alpha)\sim iqG(\beta,\alpha),\quad \partial_t G(\beta,\alpha)\sim ipG(\beta,\alpha), \tag{13}$$

and likewise for the second-order derivatives. Here, $q(\beta,\alpha)=k\cos\theta_0$ and $p(\beta,\alpha)=k\sin\theta_0$ are the normal and tangential component of the wave vector $\mathbf{k}=k[\mathbf{r}(\beta)-\mathbf{r}'(\alpha)]/|\mathbf{r}(\beta)-\mathbf{r}'(\alpha)|$ at the boundary point $\beta$ [see Fig. 1(a)].

### *2. Asymptotic form of the boundary integral kernel in momentum representation*

Following Ref. [23], we split the boundary integral into two parts; that is,

$$\int_{\mathcal{C}} d\alpha=\int_{\mathcal{C}/\Delta} d\alpha+\int_{\Delta} d\alpha,$$

where $\Delta$ refers to a small interval around $\alpha=\beta$ scaling as $\Delta\sim k^{-1+\epsilon}$ with $0<\epsilon<1$.

We deal with the short length contributions first. Due to the scaling chosen for the interval $\Delta$, we can neglect curvature contributions in the large $k$ limit and write in leading order in $1/k$,

$$\int_{\Delta} G[\mathbf{r}(\beta),\mathbf{r}'(\alpha),k]f(\alpha)d\alpha$$
$$\sim-\frac{1}{k}\int_{-k\Delta/2}^{k\Delta/2} G(0,x'/k,k)f(x'/k)dx'\sim$$
$$-\frac{1}{k}\int_{-\infty}^{\infty} G(0,x'/k,k)f(x'/k)dx', \tag{14}$$

thus integrating along a straight line in the direction of $\hat{\mathbf{t}}(\beta)$ centered at $\mathbf{r}(\beta)$. It is now convenient to express the free Green function in integral representation, which in two dimensions leads to

$$G(\mathbf{r},\mathbf{r}',k)=-\lim_{\epsilon\to 0}\int\frac{dp^2}{4\pi^2}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{p^2-(k^2+i\epsilon)}. \tag{15}$$

Aligning the $x$ axis with the tangential direction $\hat{\mathbf{t}}(\beta)$ as in Eq. (14) and integrating out the $p_y$ component, one obtains

$$G(\mathbf{r},\mathbf{r}',k)=\int\frac{dp}{2\pi}e^{ip(x-x')}\frac{e^{iq|y-y'|}}{2iq}, \tag{16}$$

with $q=\sqrt{k^2-p^2}$. Note that $\lim_{y\to 0_-}\partial_y G(\mathbf{r},\mathbf{r}')|_{y'=0}=-(1/2)\delta(x-x')$ revealing the singular behavior of the Green function in this limit.

Next, we express the single-layer distributions on the boundary in its Fourier components; that is,

$$f(\alpha)=\int dp\hat{f}(p)e^{ip\alpha}, \tag{17}$$

where we treat $p$ to leading order as a continuous variable neglecting the discreteness of $p=2\pi j/L_C$, $j\in\mathbb{N}$ due to the finite length of the boundary $\mathcal{C}$. From Eq. (16) together with Eq. (14), we obtain the short length contributions in the form

$$\int_{\Delta} G(\beta,\alpha)f(\alpha)d\alpha$$
$$\sim\frac{1}{k}\lim_{y\to 0_-}\int dpdp'e^{ip\beta}\frac{e^{iq|y|}}{2iq}\int_{-\infty}^{\infty}\frac{dx}{2\pi}e^{i(p-p')x/k}\hat{f}(p')$$
$$=\int dp\frac{\hat{f}(p)}{2iq}e^{ip\beta}. \tag{18}$$

We proceed as above for the partial derivatives (11) by identifying the normal and tangential direction with the $y$ and $x$ axis, respectively. Note again that the derivatives $\partial/\partial y$ need to be taken before completing the limit $y\to 0_-$ from below. One obtains

$$\partial_{nn}\int_{\Delta} G(\beta,\alpha)f(\alpha)d\alpha=i\int dp\hat{f}(p)\frac{q}{2}e^{ip\beta}, \tag{19}$$

$$\partial_{nt}\int_{\Delta} G(\beta,\alpha)f(\alpha)d\alpha=-i\int dp\hat{f}(p)\frac{p}{2}e^{ip\beta}, \tag{20}$$

$$\partial_{tt}\int_{\Delta} G(\beta,\alpha)f(\alpha)d\alpha=i\int dp\hat{f}(p)\frac{p^2}{2q}e^{ip\beta}. \tag{21}$$

Turning to the contributions from long trajectories, we again introduce the Green function on the boundary in terms of its Fourier components,

$$G(\beta,\alpha,k_{\mathrm{p/s}})=\int dpdp'\hat{G}^{\mathrm{p/s}}(p,p')e^{i(p\beta-p'\alpha)}, \tag{22}$$

and write

$$\int_{C/\Delta} d\alpha G(\beta,\alpha) f(\alpha) = \int dp dp' \hat{G}(p,p') \hat{f}(p') e^{ip\beta}.$$

Here, differentiation can be pulled under the integral sign and by employing the asymptotic form (12) together with a stationary phase approximation, one obtains in leading order

$$\widehat{\partial_n G}(p,p') = iq\hat{G}(p,p'), \quad \widehat{\partial_t G}(p,p') = ip\hat{G}(p,p'),$$

and likewise,

$$\widehat{\partial_{nn} G}(p,p') = -q^2\hat{G}(p,p'), \quad \widehat{\partial_{tn} G}(p,p') = -qp\hat{G}(p,p'),$$

$$\widehat{\partial_{tt} G}(p,p') = -p^2\hat{G}(p,p').$$

Writing the boundary conditions (2) in terms of the Fourier components $\hat{g}(p)$, $\hat{h}(p)$, and $G^{\mathrm{p/s}}(p,p')$, one obtains the set of equations

$$(\mathbf{M}_0 + \mathbf{M}\hat{\mathbf{D}})\hat{\mathbf{X}} = 0 \quad \text{with } \hat{\mathbf{X}} = \begin{pmatrix} \hat{g} \\ \hat{h} \end{pmatrix}, \tag{23}$$

where $\mathbf{M}_0, \mathbf{M}$ contains contributions from short and long trajectories, respectively, with

$$\mathbf{M}_0(p,p') = \frac{i}{2}\begin{pmatrix} \frac{1}{q_{\mathrm{p}}}(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) & -2\mu p \\ -2\mu p & \frac{\mu}{q_{\mathrm{s}}}(p^2 - q_{\mathrm{s}}^2) \end{pmatrix}\delta(p,p'),$$

$$\mathbf{M}(p,p') = \frac{i}{2}\begin{pmatrix} \frac{1}{q_{\mathrm{p}}}(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) & 2\mu p \\ 2\mu p & \frac{\mu}{q_{\mathrm{s}}}(p^2 - q_{\mathrm{s}}^2) \end{pmatrix}\delta(p,p'),$$

and

$$\hat{\mathbf{D}}(p,p') = 2i\begin{pmatrix} q_{\mathrm{p}}\hat{G}^{\mathrm{p}}(p,p') & 0 \\ 0 & q_{\mathrm{s}}\hat{G}^{\mathrm{s}}(p,p') \end{pmatrix} = 2\begin{pmatrix} \widehat{\partial_n G^{\mathrm{p}}}(p,p') & 0 \\ 0 & \widehat{\partial_n G^{\mathrm{s}}}(p,p') \end{pmatrix}.$$

The eigenfrequency condition for finite elastic bodies in two dimensions can thus be cast into the form

$$\det[\mathbf{I} - \hat{\mathbf{T}}(\omega)] = 0 \quad \text{with} \quad \hat{\mathbf{T}} = -\mathbf{M}_0^{-1}\mathbf{M}\hat{\mathbf{D}}, \tag{24}$$

and

$$\hat{\mathbf{T}}(p,p') = \frac{1}{4\det(\mathbf{M}_0)}\begin{pmatrix} \frac{\mu}{q_{\mathrm{s}}q_{\mathrm{p}}}(p^2 - q_{\mathrm{s}}^2)(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) + 4\mu^2 p^2 & \frac{4\mu^2 p}{q_{\mathrm{s}}}(p^2 - q_{\mathrm{s}}^2) \\ \frac{4\mu p}{q_{\mathrm{p}}}(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) & \frac{\mu}{q_{\mathrm{s}}q_{\mathrm{p}}}(p^2 - q_{\mathrm{s}}^2)(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) + 4\mu^2 p^2 \end{pmatrix}\hat{\mathbf{D}}(p,p'), \tag{25}$$

as well as

$$\det \mathbf{M}_0(p,p') = -\frac{1}{4}\left[\frac{\mu}{q_{\mathrm{s}}q_{\mathrm{p}}}(p^2 - q_{\mathrm{s}}^2)(\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2) - 4\mu^2 p^2\right]. \tag{26}$$

The operator $\hat{\mathbf{T}}$ is the short wavelength approximation of a wave propagator acting on boundary functions in Fourier or momentum representation; it has the general form of a quantum Poincaré map [17,20], written here for the elastodynamic case including mode conversion. The matrix elements $\hat{\mathbf{T}}pp'$ describe the evolution of pressure and shear waves along "ray" trajectories starting on the boundary with tangential momentum $p'$ and hitting the boundary with tangential momentum $p$; note that the rays corresponding to two different modes will in general start and end at different points on the boundary [see Fig. 1(b)]. The $q_{\mathrm{p/s}}$ component is the part of the wave vector $\mathbf{k}_{\mathrm{p/s}}$ normal to the interface and we may set

$$p_{\mathrm{p/s}} = k_{\mathrm{p/s}} \sin\theta_{\mathrm{p/s}}, \quad q_{\mathrm{p/s}} = k_{\mathrm{p/s}} \cos\theta_{\mathrm{p/s}}.$$

The tangential momentum $p$ at the end points is the same for both polarizations before and after impact with the boundary and we obtain directly Snell's law,

$$p = p_{\mathrm{p}} = k_{\mathrm{p}} \sin\theta_{\mathrm{p}} = k_{\mathrm{s}} \sin\theta_{\mathrm{s}} = p_{\mathrm{s}}. \tag{27}$$

Using $\kappa = k_{\mathrm{s}}/k_{\mathrm{p}} = c_{\mathrm{p}}/c_{\mathrm{s}}$ and identities such as

$$\lambda k_{\mathrm{p}}^2 + 2\mu q_{\mathrm{p}}^2 = (\lambda + 2\mu)k_{\mathrm{p}}^2 \cos 2\theta_{\mathrm{s}}, \quad p^2 - q_{\mathrm{s}}^2 = -k_{\mathrm{s}}^2 \cos 2\theta_{\mathrm{s}},$$

we may write the prefactor matrix in the form

$$\mathbf{A} = -\mathbf{M}_0^{-1}\mathbf{M} = \begin{pmatrix} A_{\mathrm{pp}} & A_{\mathrm{ps}} \\ A_{\mathrm{sp}} & A_{\mathrm{ss}} \end{pmatrix}, \tag{28}$$

with

$$A_{\mathrm{pp}} = A_{\mathrm{ss}} = \frac{\sin 2\theta_{\mathrm{s}} \sin 2\theta_{\mathrm{p}} - \kappa^2 \cos^2 2\theta_{\mathrm{s}}}{\sin 2\theta_{\mathrm{s}} \sin 2\theta_{\mathrm{p}} + \kappa^2 \cos^2 2\theta_{\mathrm{s}}},$$

$$A_{\mathrm{sp}} = \kappa^2 \frac{2 \sin 2\theta_{\mathrm{s}} \cos 2\theta_{\mathrm{s}}}{\sin 2\theta_{\mathrm{s}} \sin 2\theta_{\mathrm{p}} + \kappa^2 \cos^2 2\theta_{\mathrm{s}}},$$

$$A_{\mathrm{ps}} = -\frac{2\sin 2\theta_{\mathrm{p}}\cos 2\theta_{\mathrm{s}}}{\sin 2\theta_{\mathrm{s}}\sin 2\theta_{\mathrm{p}} + \kappa^2\cos^2 2\theta_{\mathrm{s}}}. \tag{29}$$

The matrix elements of $\mathbf{A}$ are up to a similarity transformation equivalent to the standard conversion factors for plane shear or pressure waves at impact with a plain interface and free boundary conditions [24]. Note that here we follow the convention used throughout the paper; for example, $A_{\mathrm{sp}}$ denotes the conversion amplitude between an incoming p wave and an outgoing s wave.

Next, we express the transition matrix $\mathbf{A}$ in a slightly different form using the transformation

$$\mathbf{a} = \mathbf{K}^{-1}\mathbf{A}\mathbf{K} \quad \text{with } \mathbf{K} = \begin{pmatrix} (q_{\mathrm{p}}/q_{\mathrm{s}})^{1/4} & 0 \\ 0 & (q_{\mathrm{s}}/q_{\mathrm{p}})^{1/4} \end{pmatrix}, \tag{30}$$

which leads to a unitary matrix $\mathbf{a}$. The relations $a_{\mathrm{pp}}^2 + a_{\mathrm{sp}}^2 = 1 = a_{\mathrm{ps}}^2 + a_{\mathrm{ss}}^2$ reflect conservation of wave energy *normal to the surface* in the presence of mode conversion [24].

#### 3. Asymptotic form of the boundary integral kernel in position representation

It is often convenient to work with the boundary integral kernel in position representation. The inverse Fourier transformation of the operator $\hat{\mathbf{T}}_{pp'} = \mathbf{A}_p\hat{\mathbf{D}}_{pp'}$ again taken in stationary phase approximation and employing the asymptotic form of the free Green function (12), yields

$$\mathbf{T}(\beta,\alpha) = \frac{1}{\sqrt{2\pi i L}}\cos\theta_0\mathbf{A}(\beta,\alpha)\begin{pmatrix} \sqrt{k_{\mathrm{p}}}e^{ik_{\mathrm{p}}L} & 0 \\ 0 & \sqrt{k_{\mathrm{s}}}e^{ik_{\mathrm{s}}L} \end{pmatrix}. \tag{31}$$

The stationary phase condition picks out contributions from shear and pressure waves traveling from $\alpha$ to $\beta$ along rays of length $L$ intersecting the boundary at $\beta$ with a common angle $\theta_0$ [see Fig. 1(a)]. In contrast to the momentum representation considered earlier, rays leaving the end point $\beta$ can do so along three different directions with angles $\theta_0$, $\theta_{\mathrm{p}}$, and $\theta_{\mathrm{s}}$. A p-polarized wave, for example, may emerge from $\beta$ at an angle $\theta_0$ or $\theta_{\mathrm{p}}$ depending on whether the corresponding incoming wave was a p or s wave. We thus set $\theta_{\mathrm{p}} \equiv \theta_0$ in $A_{\mathrm{pp}}$ and $A_{\mathrm{sp}}$ and $\theta_{\mathrm{s}} \equiv \theta_0$ in $A_{\mathrm{ps}}$ and $A_{\mathrm{ss}}$ in Eq. (29) with $\theta_{\mathrm{p}}, \theta_{\mathrm{s}}$ given by Snell's law (27); note that this implies, for example, that $A_{\mathrm{pp}} \neq A_{\mathrm{ss}}$ in general. Rewriting the operator (31) in terms of the (now in general nonunitary) transition matrix $\mathbf{a}$, one obtains

$$\mathbf{T}(\beta,\alpha) = \frac{1}{\sqrt{2\pi i L}}\begin{pmatrix} \cos\theta_0 a_{\mathrm{pp}} & \sqrt{\cos\theta_0\cos\theta_p/\kappa}\,a_{\mathrm{ps}} \\ \sqrt{\kappa\cos\theta_0\cos\theta_s}\,a_{\mathrm{sp}} & \cos\theta_0 a_{\mathrm{ss}} \end{pmatrix} \times\begin{pmatrix} \sqrt{k_{\mathrm{p}}}e^{ik_{\mathrm{p}}L} & 0 \\ 0 & \sqrt{k_{\mathrm{s}}}e^{ik_{\mathrm{s}}L} \end{pmatrix}, \tag{32}$$

with

$$a_{\mathrm{pp}} = \frac{\sin 2\theta_{\mathrm{s}}\sin 2\theta_0 - \kappa^2\cos^2 2\theta_{\mathrm{s}}}{\sin 2\theta_{\mathrm{s}}\sin 2\theta_0 + \kappa^2\cos^2 2\theta_{\mathrm{s}}},$$

$$a_{\mathrm{ps}} = -\kappa\frac{2\sqrt{\sin 2\theta_0\sin 2\theta_{\mathrm{p}}}\cos 2\theta_0}{\sin 2\theta_0\sin 2\theta_{\mathrm{p}} + \kappa^2\cos^2 2\theta_0},$$

$$a_{\mathrm{sp}} = \kappa\frac{2\sqrt{\sin 2\theta_0\sin 2\theta_{\mathrm{s}}}\cos 2\theta_{\mathrm{s}}}{\sin 2\theta_{\mathrm{s}}\sin 2\theta_0 + \kappa^2\cos^2 2\theta_{\mathrm{s}}},$$

$$a_{\mathrm{ss}} = \frac{\sin 2\theta_0\sin 2\theta_{\mathrm{p}} - \kappa^2\cos^2 2\theta_0}{\sin 2\theta_0\sin 2\theta_{\mathrm{p}} + \kappa^2\cos^2 2\theta_0}. \tag{33}$$

For hyperbolic shapes, that is, for boundaries only admitting isolated periodic geometric rays (including mode conversion at the boundary), standard arguments lead to a description of the traces of the operator $\mathbf{T}$ in terms of periodic ray trajectories [17]. One obtains

$$\mathrm{Tr}\,\mathbf{T}^n = \sum_j^{(n)} \mathcal{A}_j e^{iS_j - i\mu_j\pi/2}, \tag{34}$$

where the sum is over all periodic ray trajectories having $n$ reflections at the boundary with position and polarizations $[(\alpha_1^j, l_1^j), \ldots, (\alpha_n^j, l_n^j)]$, where $l_i^j = \mathrm{p}$ or s is the polarization of the $i$th segment of the periodic ray $j$ leaving the boundary at the point $\alpha_i$, $i = 1, \ldots, n$. Furthermore, one has

$$S_j = \sum_{i=1}^n k_{l_i^j}L_i^j, \quad \mathcal{A}_j = \mathcal{A}_j^{geo}\prod_{i=1}^n a_{l_{i+1}^j l_i^j} \tag{35}$$

taken along a periodic orbit; here $S_j$ is the action of classical mechanics and the amplitude $\mathcal{A}_j$ separates into a geometric part $\mathcal{A}_j^{geo}$ containing information about the spreading of nearby trajectories and a mode conversion loss factor. The traces $\mathbf{T}^n$ contain all the information about the spectrum and may be used to construct the density of states or express the spectral determinant (24).

The operator (32) can be written in a form more familiar from semiclassical quantum mechanics. We note that the cosine terms in the amplitudes relate to ray angles before and after hitting the boundary at $\beta$; each contribution to the periodic orbit formula (34) thus contains products of cosine terms along the periodic orbit. Following an argument by Boasman [25] developed in the scalar case, we consider

$$\sqrt{\left|\frac{\partial^2 L(\beta,\alpha)}{\partial\alpha\partial\beta}\right|} = \sqrt{\frac{\cos\theta_{\alpha\beta}\cos\theta_{\beta\alpha}}{L}}, \tag{36}$$

with angles $\theta_{\beta\alpha} = \theta_0$ taken at $\beta$ and $\theta_{\alpha\beta}$ taken at $\alpha$, respectively [see Fig. 1(a)]. The traces of the operators $\mathbf{T}$ as in Eq. (32) and $\tilde{\mathbf{T}}$ defined as

$$\tilde{\mathbf{T}}(\beta,\alpha) = \frac{1}{\sqrt{2\pi i}}\sqrt{\left|\frac{\partial^2 L_{\beta\alpha}}{\partial\alpha\partial\beta}\right|}\,\mathbf{a}(\beta,\alpha)\begin{pmatrix} \sqrt{k_{\mathrm{p}}}e^{ik_{\mathrm{p}}L_{\beta\alpha}} & 0 \\ 0 & \sqrt{k_{\mathrm{s}}}e^{ik_{\mathrm{s}}L_{\beta\alpha}} \end{pmatrix} \tag{37}$$

are thus equivalent to leading order. That is, when writing the traces as a sum over periodic rays as in Eq. (34), the cosine terms coincide after multiplication along a periodic orbit. Similarly, the extra $\kappa^{\pm 1/2}$ terms in the off-diagonal terms in Eq. (32) cancel after one period. This confirms the form of the operator as postulated in Ref. [22] from which the trace formula suggested by Couchman *et al.* [21] can be derived by standard means as indicated earlier.

## III. CONCLUSION

We have derived an asymptotic form of the boundary integral kernel in 2D elastodynamics from which periodic orbit trace formulas can be deduced using stationary phase arguments. It is expected that a 3D version of the asymptotic operator can be written in the form (37) using local coordinates where the tangential direction lies in the plane spanned by the vector $\mathbf{r}-\mathbf{r}'$ and the normal at the boundary point $\mathbf{r}$. In deriving the 3D version of the operator (37) one is naturally lead to a momentum representation in terms of spherical coordinates; the technical difficulties are not expected to exceed those of the 3D quantum case as discussed in Refs. [17,19,20].

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[1] C. Draeger and M. Fink, Phys. Rev. Lett. **79**, 407 (1997); M. Fink, D. Cassereau, A. Derode, C. Prada, P. Roux, M. Tanter, J.-L. Thomas, and F. Wu, Rep. Prog. Phys. **63**, 1933 (2000); A. Tourin, F. Van Der Biest, and M. Fink, Phys. Rev. Lett. **96**, 104301 (2006).
[2] R. L. Weaver and O. I. Lobkis, Phys. Rev. Lett. **84**, 4942 (2000); J. de Rosny, A. Tourin, and M. Fink, *ibid.* **84**, 1693 (2000).
[3] M. G. Brown, J. A. Colosi, S. Tomsovic, A. L. Virovlyansky, M. A. Wolfson, and G. M. Zaslavsky, J. Acoust. Soc. Am. **113**, 2533 (2003); F. J. Beron-Vera, M. G. Brown, J. A. Colosi, S. Tomsovic, A. L. Virovlyansky, M. A. Wolfson, and G. M. Zaslavsky, *ibid.* **113**, 1226 (2003).
[4] R. H. Lyon and R. J. DeJong, *Theory and Application of Statistical Energy Analysis*, 2nd ed. (Butterworth-Heinemann, Boston, MA, 1995).
[5] H. Sato and M. Fehler, *Seismic Wave Propagation and Scattering in the Heterogenous Earth* (Springer-Verlag, Heidelberg, 1998).
[6] E. Larose, L. Margerin, B. A. van Tiggelen, and M. Campillo, Phys. Rev. Lett. **93**, 048501 (2004).
[7] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic Press, London, 1991).
[8] U. Kuhl, H.-J. Stoeckmann, and R. Weaver, J. Phys. A **38**, 10433 (2005).
[9] R. L. Weaver and O. I. Lobkis, Phys. Rev. Lett. **87**, 134301 (2001); J. Acoust. Soc. Am. **116**, 2731 (2004).
[10] M. Campillo and A. Paul, Science **299**, 547 (2003).
[11] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990).
[12] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press, Cambridge, UK, 1999).
[13] G. Tanner, K. Richter, and J.-M. Rost, Rev. Mod. Phys. **72**, 497 (2000).
[14] M. Kitahara, *Boundary Integral Equation Methods in Eigenvalue Problems of Elastodynamics and Thin Plates* (Elsevier, Amsterdam, 1985).
[15] M. Bonnet, *Boundary Integral Equation Methods for Solids and Fluids* (John Wiley and Sons, Chichester, 1995).
[16] B. R. Mace, J. Sound Vib. **264**, 391 (2003); **279**, 141 (2005).
[17] E. Bogomolny, Nonlinearity **5**, 805 (1992).
[18] E. Doron and U. Smilansky, Nonlinearity **5**, 1055 (1992).
[19] C. Rouvinez and U. Smilansky, J. Phys. A **28**, 77 (1995).
[20] T. Prosen, J. Phys. A **27**, L709 (1994); **28**, 4133 (1995); Physica D **91**, 244 (1996).
[21] L. Couchman, E. Ott, and T. M. Antonsen, Jr., Phys. Rev. A **46**, 6193 (1992).
[22] N. Søndergaard and G. Tanner, Phys. Rev. E **66**, 066211 (2002).
[23] E. Bogomolny and E. Hugues, Phys. Rev. E **57**, 5404 (1998).
[24] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon, Oxford, 1959).
[25] P. A. Boasman, Nonlinearity **7**, 485 (1994).